Numerical Relativity on CMC Hypersurfaces

Luisa T. Buchman
ACCGR, Brown, May 2011

with
James Bardeen
Harald Pfeiffer
Olivier Sarbach

Outline

- Purpose:

  Evolve binary black holes on constant mean curvature (CMC) hypersurfaces which reach future null infinity ($\mathscr{I}^+$), for high-accuracy gravitational wave modeling.

- Results:

  - Initial Data (Bowen-York)
  - Bondi-Sachs Energy-Momentum
  - Evolution scheme

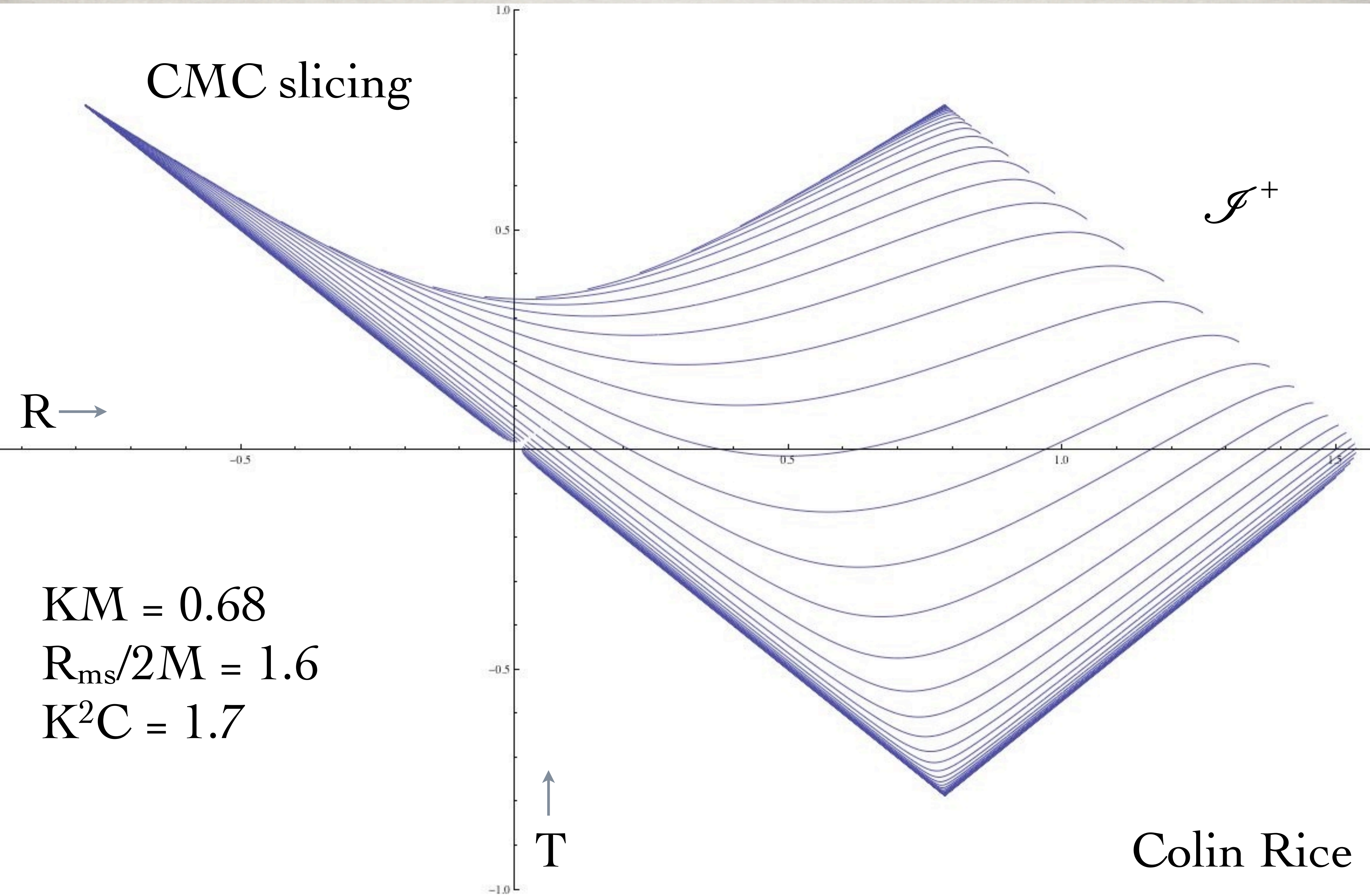
Why $\mathscr{I}^+$?

- To compute highly accurate waveforms without losing computational efficiency.
- Gravitational radiation is well-defined at $\mathscr{I}^+$ (*Bondi et al. 1962; Sachs 1962)*. Bondi news contains all the gravitational wave information.
- No approximate wave extraction at finite radii.
- No approximate boundary conditions on a truncated domain.
- Conformal compactification gives smaller computational grids.

Why CMC?

- Cauchy characteristic matching / extraction (*see Winicour, Living Rev. Relativity, 2009 / Reisswig et al., 2010*).
- Constant mean curvature (CMC):
    - ➡ a simple class of hyperboloidal slicing (TrK = const.)
    - ➡ constraint equations partially decoupled
    - ➡ compatible with conformal compactification
    - ➡ match conventional slicing near black holes
    - ➡ smoothly become asymptotically null

Penrose diagram: Schwarzschild spacetime



$\mathscr{I}^+ \longrightarrow$ FINITE COORDINATE RADIUS

- Finite numerical grid extending all the way to $\mathscr{I}^+$.
- Problem: compactified spatial coordinates & asymptotically null spatial hypersurfaces

  ➡ physical spacetime metric $g_{\mu\nu}$ singular at $\mathscr{I}^+$.

- Solution: conformal approach (*Penrose 1964*)

  ➡ conformal factor: $\Omega\big|_{\mathscr{I}^+} = 0$,

  ➡ conformal metric: $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ (regular at $\mathscr{I}^+$)

Initial Data

- Hyperboloidal Bowen-York (conformally flat) data on CMC hypersurfaces *(Buchman, Pfeiffer, Bardeen 2009)*.

  Hamiltonian constraint:

$$\tilde{\nabla}^2\Omega - \frac{3}{2\Omega}\tilde{\nabla}_k\Omega\tilde{\nabla}^k\Omega + \frac{\Omega}{4}\tilde{\mathcal{R}} + \frac{(TrK)^2}{6\Omega} - \frac{\Omega^5}{4}\tilde{A}^{ij}\tilde{A}_{ij} = 0$$

next slide

  ➡ $\tilde{A}_{ij}$ is obtained from known Bowen-York solutions to the momentum constraint: $\tilde{\nabla}_j\tilde{A}^{ij} = 0$

  ➡ elliptic equation, singular at $\mathscr{I}^+$ ($\Omega|_{\mathscr{I}^+} = 0$).

  ➡ forces $\left(\tilde{\nabla}_k\Omega\tilde{\nabla}^k\Omega\right)_{\mathscr{I}^+} = \left(\frac{\mathrm{TrK}}{3}\right)^2$

- Pfeiffer elliptic solver (Caltech-Cornell-CITA SpEC code)

  ➡ no special handling of singular terms at $\mathscr{I}^+$.

Generalized Bowen-York solution

Momentum constraint: $\tilde{\nabla}_j \tilde{A}^{ij} = 0$

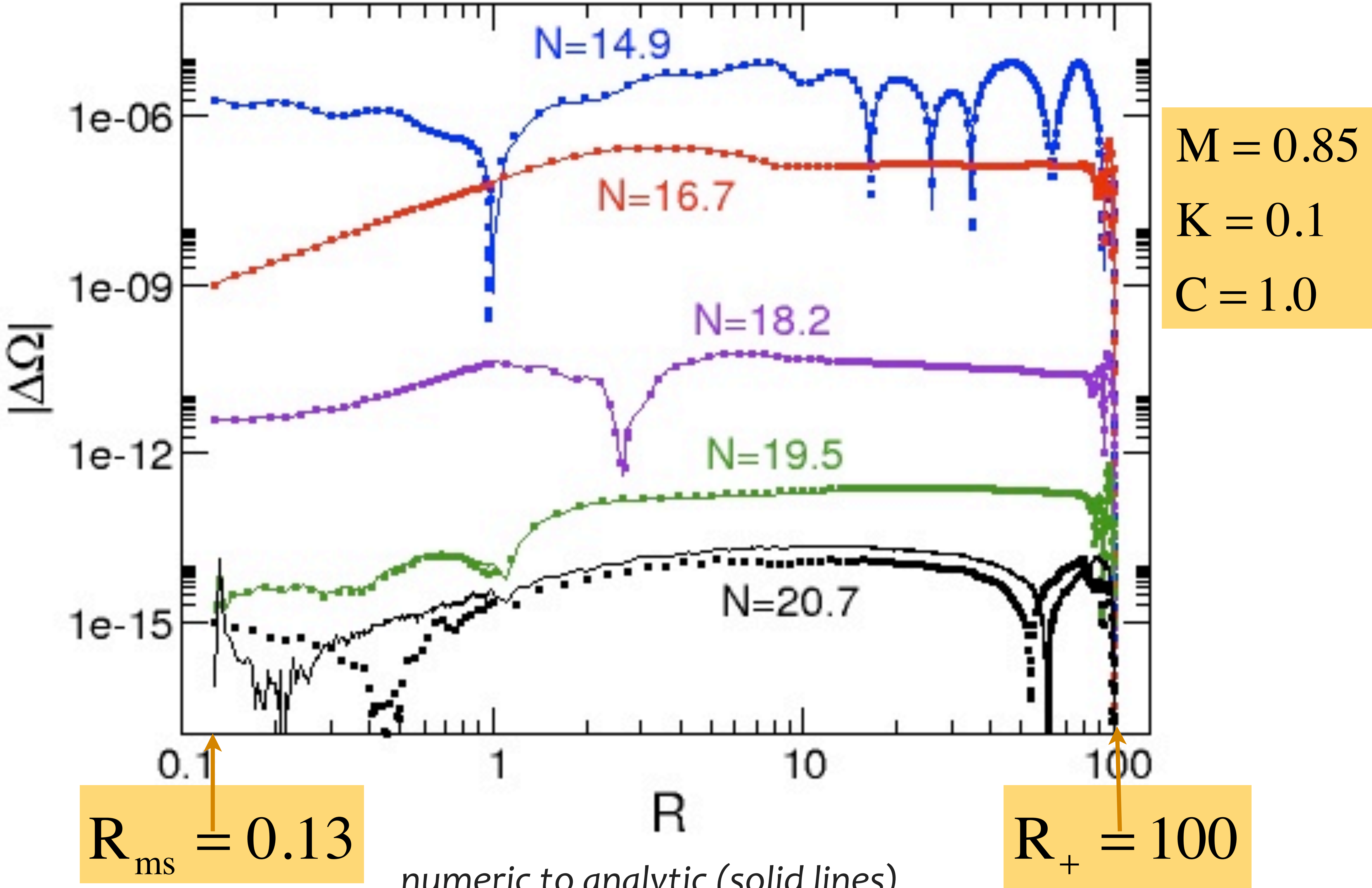
$$\tilde{A}_{ij} = \frac{C}{R^3}\left(3n_i n_j - \delta_{ij}\right) \quad \longleftarrow \text{Schwarzschild}$$

$$-\frac{3}{2R^2}\left[P_i n_j + P_j n_i + P^k n_k\left(n_i n_j - \delta_{ij}\right)\right] \longleftarrow \text{boost upper}$$

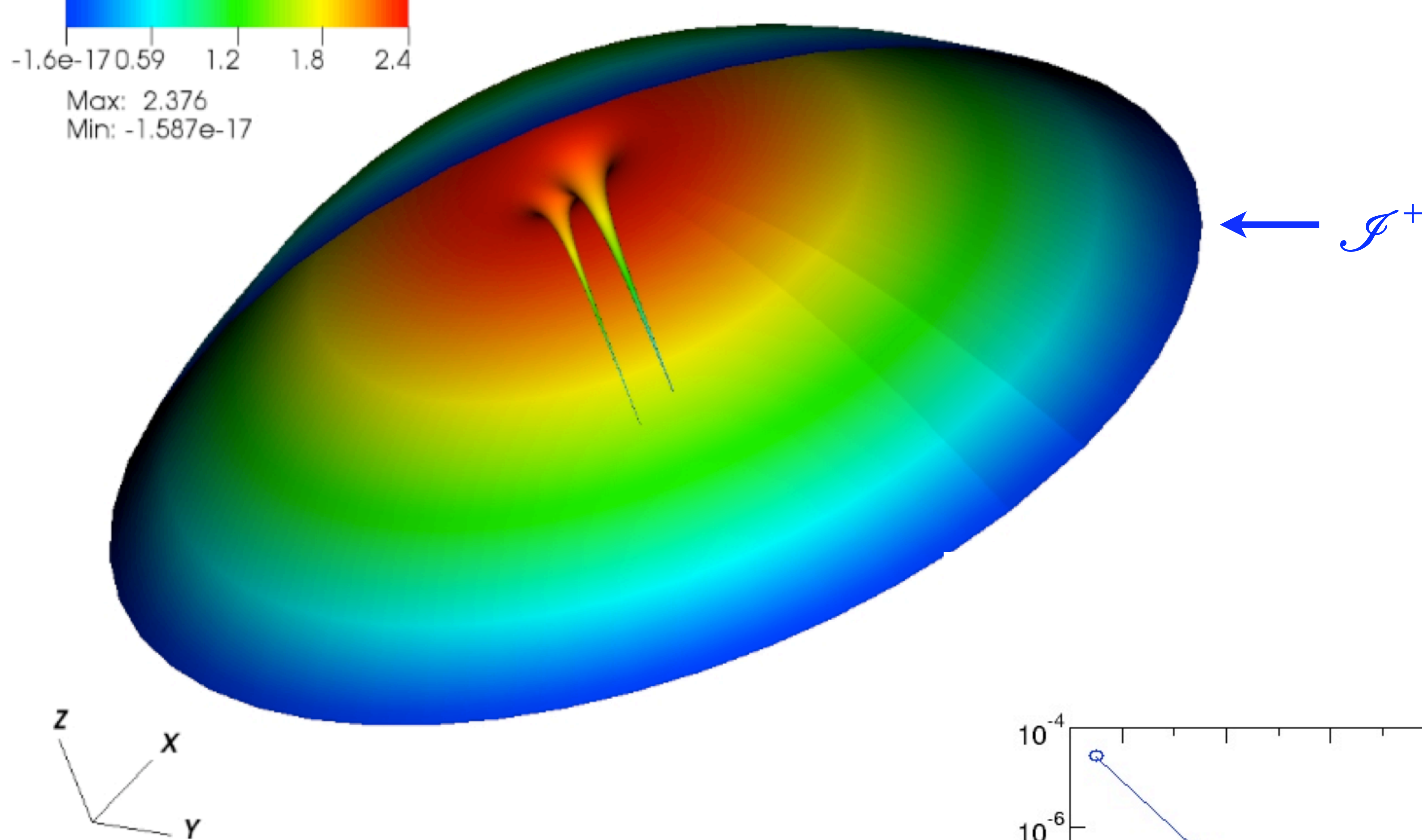
$$-\frac{3}{R^3}\left[\varepsilon_{ik\ell} S^k n^\ell n_j + \varepsilon_{jk\ell} S^k n^\ell n_i\right] \longleftarrow \text{spin}$$

$$+\frac{3}{2R^4}\left[Q_i n_j + Q_j n_i + Q^k n_k\left(-5 n_i n_j + \delta_{ij}\right)\right] \longleftarrow \text{boost lower}$$

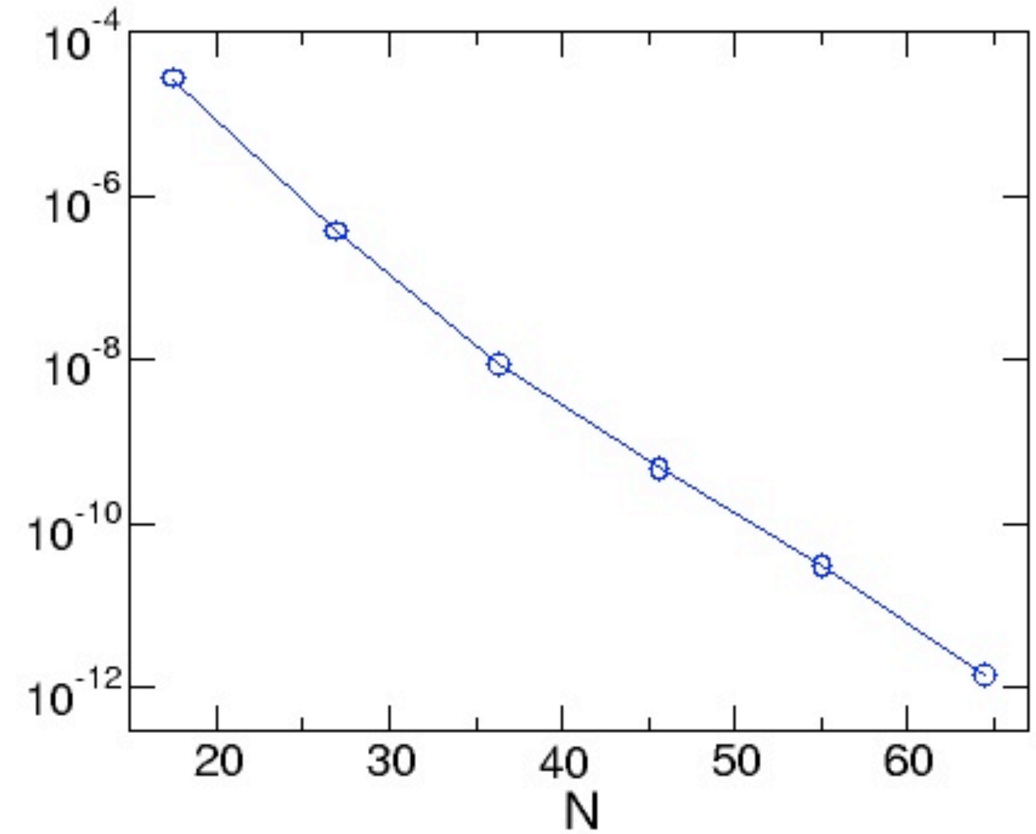
Schwarzschild Convergence Study: Ω



numeric to analytic (solid lines)
numeric truncation error (dotted lines)



Conformal factor for mass ratio 2:1 boosted spinning BBH



Bondi-Sachs Energy & Momentum

- True physical quantities defined at $\mathscr{I}^+$ for asymptotically flat spacetimes using retarded null coordinates *(Bondi, van der Burg, Metzner 1962; Sachs 1962)*

Bondi-Sachs Mass aspect M_A:

- Monopole moment of M_A gives Energy E_{BS}
- Dipole moment of M_A gives Linear Momentum $\vec{P}_{BS}$

- CMC slicing various methods, e.g. *Chruściel, Jezierski, and Leski 2004*

Our approach *(Bardeen and Buchman 2011 --in prep.)*:

- $\dot{\mathscr{I}}^+$ is the intersection of CMC slice with $\mathscr{I}^+$

For now, focus on the conformally flat case

- => $\dot{\mathscr{I}}^+$ is a coordinate sphere with radius R_+

Bondi-Sachs mass aspect

$$M_{\mathrm{A}} = -\frac{3}{K}\left[4c_3 + \frac{d_1}{2}\right] = M_{\Omega} + M_{\mathrm{K}}$$

➡ c_3 coefficient in the asymptotic expansion of Ω away from $\mathscr{I}^+$ (obtain Ω by solving the Hamiltonian constraint)

➡ $d_1 \equiv \frac{K^2}{9} R_+^3 \tilde{A}_{ij} N^i N^j$ (analytic; from Bowen-York solution)

↖ N^k radial unit vector

- Additionally, for the conformally flat case, there is a deceptively simple formula for the angular momentum:

$$\vec{J} = \vec{S} + \vec{D} \times \vec{P}$$

Boosted Bowen-York Black Hole

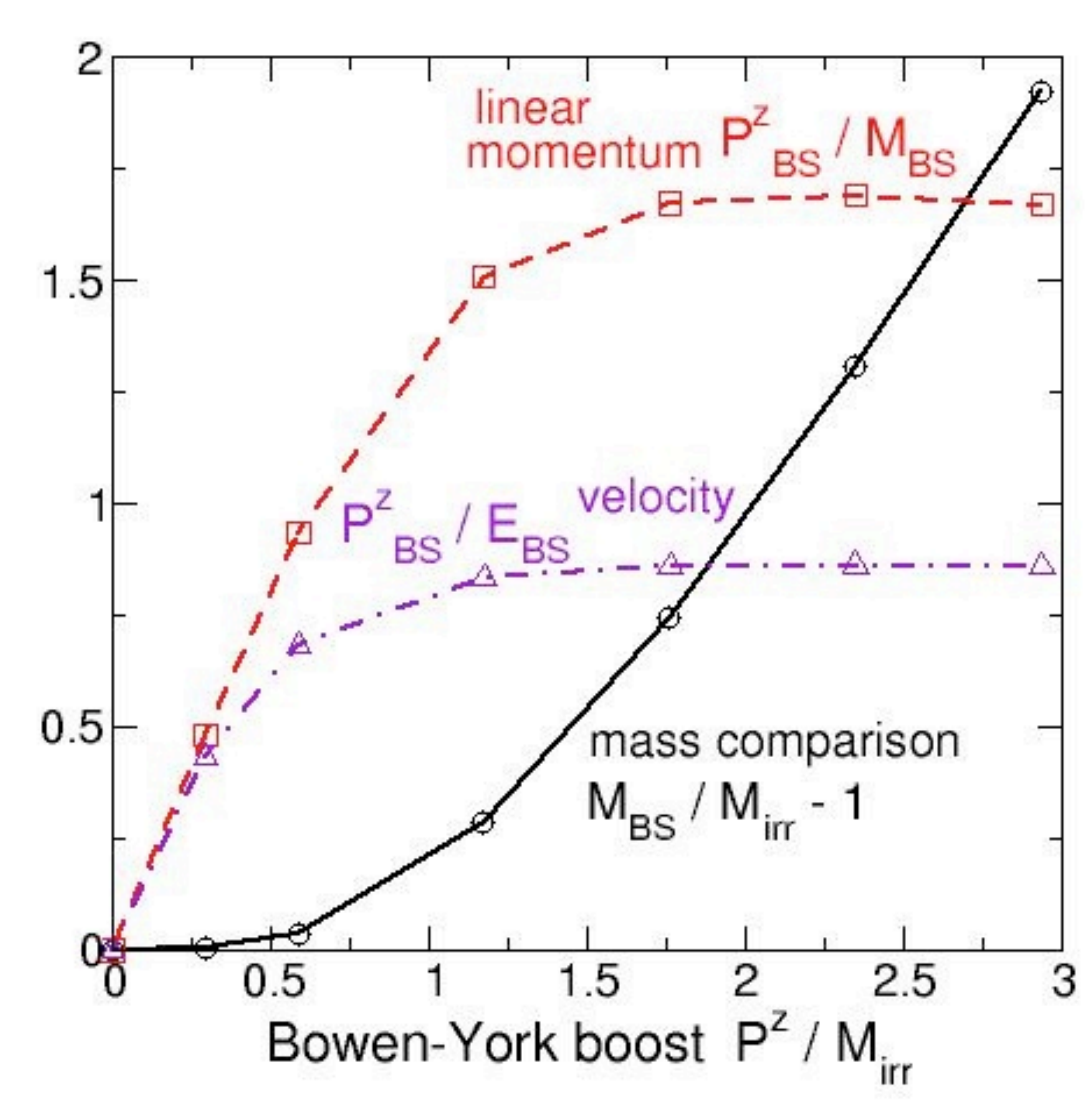
- Non-spinning, centered.
- Bowen-York boost P^z .
- Irreducible mass:

$$M_{irr} = \sqrt{A/16\pi}$$

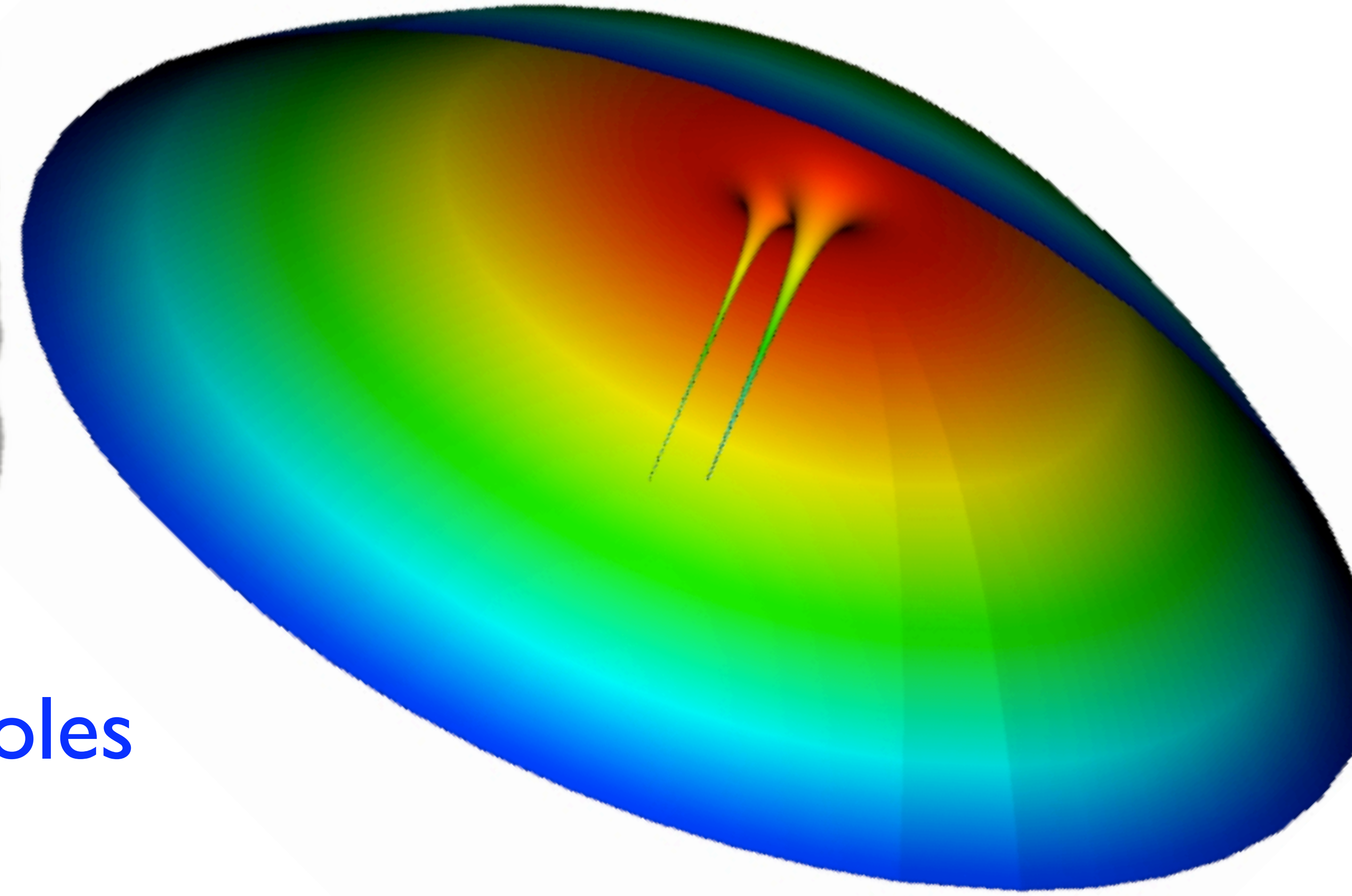
- Bondi-Sachs mass:

$$M_{\mathrm{BS}} = \sqrt{\left(E_{\mathrm{BS}}\right)^2 - \left(\vec{P}_{\mathrm{BS}}\right)^2}$$

- CMC slice: Bondi-Sachs E_{BS}, P_{BS} and M_{BS} are the physical quantities, not P^z and M_{irr}.



Bondi-Sachs physical quantities for BBH initial data



$R_+ = 100$ ← location of $\mathscr{I}^+$

$D = 12$ ← distance between holes

$M_{irr}(\text{hole A}) = 0.53$

$M_{irr}(\text{hole B}) = 0.27$

$M_{irr} = 0.80$

$M_{BS} = 0.98$ ← physical mass

$\vec{S}\ (\text{hole A}) = 0.4\hat{y}$

$\vec{S}\ (\text{hole B}) = -0.1\hat{y}$

$\vec{J} = 0.3\hat{y} + 0.7\hat{z}$ ← physical angular momentum

$\vec{P}_{BS} = 0$ ← total physical linear momentum

$R_+ = 33.3$

results same

20% less wall-clock time

CONFORMAL EVOLUTION ON HYPERBOLOIDAL SLICES

- Friedrich 1983 hyperbolic, manifestly regular, tetrad (Weyl tensor)
  - numerical (*Frauendiener review 2004*).
- Zenginoğlu 2008 hyperbolic, generalized harmonic, metric-based
  - numerical *(Zenginoğlu and Tiglio 2009; Zenginoğlu and Kidder 2010)*
- Moncrief & Rinne 2009 hyperbolic-elliptic, metric-based, CMC
  - numerical (*Rinne 2010)* long-term stable dynamical Einstein evolution in axisymmetry
- Bardeen, Sarbach & Buchman 2011 hyperbolic-elliptic, tetrad, CMC (BSB scheme)

CONFORMAL EVOLUTION: BSB SCHEME

Tetrad variables: 24 connection coefficients à la *Estabrook and Wahlquist 1964*

DYNAMIC

$\hat{\tilde{K}}_{ab}$ ← traceless part of the conformal extrinsic curvature, $\tilde{K}_{ab}$ (5)

$\hat{\tilde{N}}_{ab}$ ← symmetric traceless part of $\tilde{N}_{ab}$, which is the dyadic form of the spatial conformal connection coefficients (5)

$\tilde{B}_a{}^k$ ← coordinate components of conformal spatial triad vectors (9)

GAUGE

$\tilde{K}, \tilde{N} \to 0$ ← traces of $\tilde{K}_{ab}, \tilde{N}_{ab}$ respectively (1)

$\tilde{n}_b \to 0$ ← antisymmetric part of $\tilde{N}_{ab}$

$\tilde{\omega}_b$ ← conformal angular velocity wrt Fermi Walker transport (3)

$\tilde{a}_b$ ← conformal acceleration of tetrad frame

$\tilde{\alpha}$ ← conformal lapse (1)

Ω ← conformal factor (1)

β^k ← coordinate components of shift vector (3)

Note: all variables are scalar fields on CMC slice except $\tilde{B}_a{}^k$ and β^k

Gauge conditions

- Hypersurface-orthogonal fixes tetrad boost freedom: $\tilde{a}_b = \tilde{B}_a{}^k \partial_k \log \tilde{\alpha}$

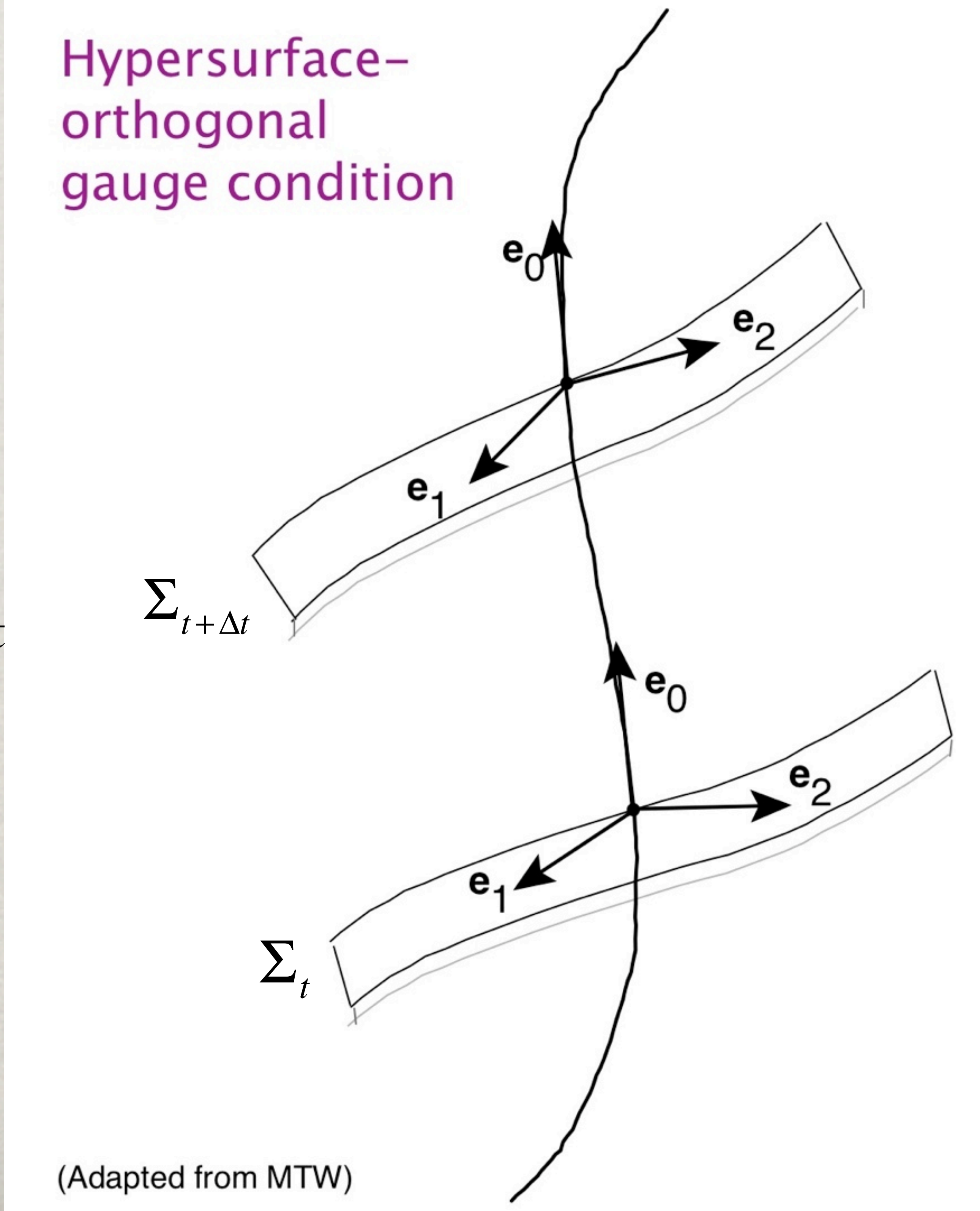
- CMC slicing fixes $\tilde{\alpha}$ *elliptic*

- 3D Nester gauge motivates choice of conformal gauge and fixes tetrad rotational freedom, so that $\tilde{n}_b = 0,\ \tilde{N} = 0$

- Ω determined by Hamiltonian constraint *elliptic*

- Preservation of 3D Nester gauge in time gives elliptic equatons for $\tilde{\omega}_b,\ \tilde{K}$ *elliptic*

- Shift: several alternatives *elliptic*



Evolution Equations

Note:
$\tilde{D}_0 = \frac{1}{\tilde{\alpha}}\left(\partial_t - \mathcal{L}_\beta\right),\ \tilde{D}_a = \tilde{B}_a^{\ k}\partial_k$

- Maxwell-like:

$$\tilde{D}_0 \hat{\tilde{N}}_{ab} + \tilde{D}_c \hat{\tilde{K}}_{d(a}\varepsilon_{b)cd} = \ldots$$

$$\tilde{D}_0 \hat{\tilde{K}}_{ab} - \tilde{D}_c \hat{\tilde{N}}_{d(a}\varepsilon_{b)cd} = \ldots$$

- Plus advection equation for conformal triad vectors

$$\tilde{D}_0 \tilde{B}_a^k = \ldots$$

- 19 total evolution equations

Regularity Conditions

- Singularities at $\mathscr{I}^+$ in equations for:

  - $\Omega,\ \tilde{\alpha}$ Singularities in these elliptic equations force the solutions to have particular asymptotic behaviors (recall example on slide 7).

  - $\hat{\tilde{K}}_{ab}$ Singularity in source term of this evolution equation is finite with the following conditions:

    1. $\hat{\tilde{\kappa}}_{ab} \doteq \hat{\tilde{K}}_{ab}$, where $\hat{\tilde{\kappa}}_{ab}$ is the 2D traceless extrinsic curvature of $\dot{\mathscr{I}}^+$. This is the zero-shear condition.

    2. Penrose regularity condition, that the conformally invariant Weyl tensor vanish at $\mathscr{I}^+$.

  - Once imposed in the initial conditions, these regularity conditions are preserved by the evolution equations.

Boundary Conditions at $\mathscr{I}^+$

- None required for the evolution equations.
- BCs on the elliptic equations -- an advantage!
  - BC on shift equation ensures:
    1. R_+ is kept at a fixed coordinate radius,
    2. the angular coordinates of $\dot{\mathscr{I}}^+$ are propagated along the null generators of $\mathscr{I}^+$,
    3. a simple relation between the computational coordinates and standard polar coordinates on $\dot{\mathscr{I}}^+$.
  - BC on $\tilde{K}$ keeps the intrinsic geometry of $\dot{\mathscr{I}}^+$ a 2-sphere with constant area $4\pi\xi_0^{-2}$. This implies that the expansion of $\mathscr{I}^+$ vanishes.
  - BC on $\tilde{\alpha}$ makes the time coordinate correspond to retarded Minkowski time at $\mathscr{I}^+$.
- Because of the above properties, we obtain a simple expression for Bondi news function in terms of our variables (next slide):

Bondi News!

$$\frac{3}{K\xi_0}\breve{\chi}_{AB}$$

- $\breve{\chi}_{AB}$ is the asymptotic gravitational wave amplitude.

- It is equal to the traceless part of the 2D extrinsic curvature of $\dot{\mathscr{I}}^+$, which is a tensor that can be calculated in any coordinate system. Note that (A,B) are the angular coordinates on the 2-sphere $\dot{\mathscr{I}}^+$.

Summary

- Seek to evolve binary black holes on CMC hypersurfaces which reach future null infinity ($\mathscr{I}^+$) for high-accuracy gravitational wave modeling.

- Results presented:

  i) Generalized hyperboloidal Bowen-York binary black hole initial data.

  ii) Bondi-Sachs energy and momentum for data on CMC slices (limited presentation to Bowen-York data).

  iii) Evolution scheme.