Numerical Relativity on CMC Hypersurfaces

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with
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Purpose:

Evolve binary black holes on constant mean curvature (CMC) hypersurfaces which reach future null infinity ($\mathcal{I}^+$), for high-accuracy gravitational wave modeling.

Results:

- Initial Data (Bowen-York)
- Bondi-Sachs Energy-Momentum
- Evolution scheme
To compute highly accurate waveforms without losing computational efficiency.

Gravitational radiation is well-defined at $I^+$ (Bondi et al. 1962; Sachs 1962). Bondi news contains all the gravitational wave information.

No approximate wave extraction at finite radii.

No approximate boundary conditions on a truncated domain.

Conformal compactification gives smaller computational grids.
**Why CMC?**

- Cauchy characteristic matching / extraction (see Winicour, *Living Rev. Relativity*, 2009 / Reisswig et al., 2010).

- Constant mean curvature (CMC):
  - a simple class of hyperboloidal slicing ($\text{Tr}K = \text{const.}$)
  - constraint equations partially decoupled
  - compatible with conformal compactification
  - match conventional slicing near black holes
  - smoothly become asymptotically null
Penrose diagram: Schwarzschild spacetime

$R \rightarrow$

CMC slicing

$KM = 0.68$

$R_{ms}/2M = 1.6$

$K^2C = 1.7$

Colin Rice
Finite numerical grid extending all the way to \( J^+ \).

Problem: compactified spatial coordinates & asymptotically null spatial hypersurfaces

\[ \text{physical spacetime metric} \quad g_{\mu\nu} \text{ singular at } J^+. \]

Solution: conformal approach (Penrose 1964)

\[ \text{conformal factor:} \quad \Omega|_{J^+} = 0, \]

\[ \text{conformal metric:} \quad \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \text{ (regular at } J^+). \]
Hyperboloidal Bowen-York (conformally flat) data on CMC hypersurfaces (Buchman, Pfeiffer, Bardeen 2009).

**Hamiltonian constraint:**

\[
\tilde{\nabla}^2 \Omega - \frac{3}{2\Omega} \tilde{\nabla}_k \Omega \tilde{\nabla}^k \Omega + \frac{\Omega}{4} \bar{\mathcal{R}} + \frac{(TrK)^2}{6\Omega} - \frac{\Omega^{5}}{4} \tilde{A}^{ij} \tilde{A}_{ij} = 0
\]

- \( \tilde{A}_{ij} \) is obtained from known Bowen-York solutions to the momentum constraint: \( \tilde{\nabla}_j \tilde{A}^{ij} = 0 \)
- elliptic equation, singular at \( \mathcal{I}^+ \) ( \( \Omega|_{\mathcal{I}^+} = 0 \) ).
- forces \( (\tilde{\nabla}_k \Omega \tilde{\nabla}^k \Omega)_{\mathcal{I}^+} = \left(\frac{TrK}{3}\right)^2 \)
- Pfeiffer elliptic solver (Caltech-Cornell-CITA SpEC code)
- no special handling of singular terms at \( \mathcal{I}^+ \).
Momentum constraint: $\nabla_j \tilde{A}^{ij} = 0$

$\tilde{A}_{ij} = \frac{C}{R^3} (3n_i n_j - \delta_{ij})$

$-\frac{3}{2R^2} \left[ P_i n_j + P_j n_i + P^k n_k \left( n_i n_j - \delta_{ij} \right) \right]$  \(\text{Schwarzschild}\)

$-\frac{3}{R^3} \left[ \epsilon_{ik\ell} S^k n_j + \epsilon_{jk\ell} S^k n_i \right]$  \(\text{spin}\)

$+ \frac{3}{2R^4} \left[ Q_i n_j + Q_j n_i + Q^k n_k \left( -5n_i n_j + \delta_{ij} \right) \right]$  \(\text{boost lower}\)$
Schwarzschild Convergence Study: \( \Omega \)

- \( R_{\text{rms}} = 0.13 \)
- \( R_+ = 100 \)
- Numeric to analytic (solid lines)
- Numeric truncation error (dotted lines)

Parameters:
- \( M = 0.85 \)
- \( K = 0.1 \)
- \( C = 1.0 \)
Conformal factor for mass ratio 2:1 boosted spinning BBH
True physical quantities defined at $\mathcal{I}^+$ for asymptotically flat spacetimes using retarded null coordinates (Bondi, van der Burg, Metzner 1962; Sachs 1962)

**Bondi-Sachs Mass aspect $M_A$:**

- Monopole moment of $M_A$ gives Energy $E_{BS}$
- Dipole moment of $M_A$ gives Linear Momentum $P_{BS}$

**CMC slicing various methods,** e.g. Chruściel, Jezierski, and Leski 2004

**Our approach** (Bardeen and Buchman 2011 --in prep.):

- $\mathcal{I}^+$ is the intersection of CMC slice with $\mathcal{I}^+$
- For now, focus on the conformally flat case
- $\Rightarrow \mathcal{I}^+$ is a coordinate sphere with radius $R_+$
Bondi-Sachs mass aspect

\[ M_A = -\frac{3}{K} \left[ 4c_3 + \frac{d_1}{2} \right] = M_\Omega + M_K \]

- \( c_3 \) coefficient in the asymptotic expansion of \( \Omega \) away from \( \mathcal{I}^+ \) (obtain \( \Omega \) by solving the Hamiltonian constraint)

\[ d_1 \equiv \frac{K^2}{9} R^3_+ \tilde{A}_{ij} N^i N^j \] (analytic; from Bowen-York solution)

- \( \Omega \) radial unit vector

Additionally, for the conformally flat case, there is a deceptively simple formula for the angular momentum:

\[ \vec{J} = \vec{S} + \vec{D} \times \vec{P} \]
• Non-spinning, centered.

• Bowen-York boost $P^z$.

• Irreducible mass:

$$M_{irr} = \sqrt{A / 16\pi}$$

• Bondi-Sachs mass:

$$M_{BS} = \sqrt{(E_{BS})^2 - (\vec{P}_{BS})^2}$$

• CMC slice: Bondi-Sachs $E_{BS}$, $P_{BS}$ and $M_{BS}$ are the physical quantities, not $P^z$ and $M_{irr}$. 
\( R_+ = 100 \)  \( \leftarrow \) location of \( \mathcal{I}^+ \)
\( D = 12 \)  \( \leftarrow \) distance between holes
\( M_{\text{irr}} \) (hole A) = 0.53
\( M_{\text{irr}} \) (hole B) = 0.27
\( M_{\text{irr}} \) = 0.80
\( M_{\text{BS}} \) = 0.98  \( \leftarrow \) physical mass
\( \mathbf{\hat{S}} \) (hole A) = 0.4\( \mathbf{\hat{y}} \)
\( \mathbf{\hat{S}} \) (hole B) = -0.1\( \mathbf{\hat{y}} \)
\( \mathbf{\hat{J}} \) = 0.3\( \mathbf{\hat{y}} \) + 0.7\( \mathbf{\hat{z}} \)  \( \leftarrow \) physical angular momentum
\( \mathbf{\hat{P}}_{\text{BS}} = 0 \)  \( \leftarrow \) total physical linear momentum

\( R_+ = 33.3 \)
results same
20% less
wall-clock
 time
Conformal evolution on hyperboloidal slices

Friedrich 1983 hyperbolic, manifestly regular, tetrad (Weyl tensor)
• numerical (Frauendiener review 2004).

Zenginoğlu 2008 hyperbolic, generalized harmonic, metric-based
• numerical (Zenginoğlu and Tiglio 2009; Zenginoğlu and Kidder 2010)

Moncrief & Rinne 2009 hyperbolic-elliptic, metric-based, CMC
• numerical (Rinne 2010) long-term stable dynamical Einstein evolution in axisymmetry

Bardeen, Sarbach & Buchman 2011 hyperbolic-elliptic, tetrad, CMC (BSB scheme)
**Conformal Evolution: BSB Scheme**

Tetrad variables: 24 connection coefficients à la *Estabrook and Wahlquist 1964*

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{K}_{ab}$</td>
<td>traceless part of the conformal extrinsic curvature, $\hat{K}_{ab}$ (5)</td>
</tr>
<tr>
<td>$\tilde{N}_{ab}$</td>
<td>symmetric traceless part of $\tilde{N}_{ab}$, which is the dyadic form of the spatial conformal connection coefficients (5)</td>
</tr>
<tr>
<td>$\tilde{B}^k_a$</td>
<td>coordinate components of conformal spatial triad vectors (9)</td>
</tr>
<tr>
<td>$\tilde{K}, \tilde{N}$</td>
<td>traces of $\hat{K}<em>{ab}, \tilde{N}</em>{ab}$ respectively (1)</td>
</tr>
<tr>
<td>$\tilde{\omega}_b$</td>
<td>antisymmetric part of $\tilde{N}_{ab}$</td>
</tr>
<tr>
<td>$\tilde{a}_b$</td>
<td>conformal angular velocity wrt Fermi Walker transport (3)</td>
</tr>
<tr>
<td>$\tilde{\alpha}$</td>
<td>conformal acceleration of tetrad frame</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>conformal lapse (1)</td>
</tr>
<tr>
<td>$\beta^k$</td>
<td>coordinate components of shift vector (3)</td>
</tr>
</tbody>
</table>

**Note:** all variables are scalar fields on CMC slice except $\tilde{B}^k_a$ and $\beta^k$
Hypersurface-orthogonal fixes tetrad boost freedom:
\[ \tilde{a}_b = \tilde{B}^k_a \partial_k \log \tilde{\alpha} \]

CMC slicing fixes \( \tilde{\alpha} \) **elliptic**

3D Nester gauge motivates choice of conformal gauge and fixes tetrad rotational freedom, so that
\[ \tilde{n}_b = 0, \tilde{N} = 0 \]

**\( \Omega \)** determined by Hamiltonian constraint **elliptic**

Preservation of 3D Nester gauge in time gives elliptic equations for \( \tilde{\omega}_b, \tilde{K} \) **elliptic**

Shift: several alternatives **elliptic**

(Adapted from MTW)
Evolution Equations

Maxwell-like: \[ \tilde{D}_0 \hat{N}_{ab} + \tilde{D}_c \hat{K}_{d(a} \varepsilon_{b)cd} = \ldots \]
\[ \tilde{D}_0 \hat{K}_{ab} - \tilde{D}_c \hat{N}_{d(a} \varepsilon_{b)cd} = \ldots \]

Plus advection equation for conformal triad vectors

\[ \tilde{D}_0 \hat{B}_a^k = \ldots \]

19 total evolution equations

Note:
\[ \tilde{D}_0 = \frac{1}{\tilde{\alpha}} (\partial_t - \mathcal{L}_\beta), \quad \tilde{D}_a = \tilde{B}_a^k \partial_k \]
Singularities at $\mathcal{I}^+$ in equations for:

1. $\Omega, \tilde{\alpha}$  Singularities in these elliptic equations force the solutions to have particular asymptotic behaviors (recall example on slide 7).
2. $\hat{K}_{ab}$  Singularity in source term of this evolution equation is finite with the following conditions:
   1. $\hat{K}_{ab} = \hat{K}_{ab}$, where $\hat{K}_{ab}$ is the 2D traceless extrinsic curvature of $\mathcal{I}^+$. This is the zero-shear condition.
   2. Penrose regularity condition, that the conformally invariant Weyl tensor vanish at $\mathcal{I}^+$.

• Once imposed in the initial conditions, these regularity conditions are preserved by the evolution equations.
None required for the evolution equations.

BCs on the elliptic equations -- an advantage!

- BC on shift equation ensures:
  1. $R_+$ is kept at a fixed coordinate radius,
  2. the angular coordinates of $\mathcal{I}^+$ are propagated along the null generators of $\mathcal{I}^+$,
  3. a simple relation between the computational coordinates and standard polar coordinates on $\mathcal{I}^+$.

- BC on $\tilde{K}$ keeps the intrinsic geometry of $\mathcal{I}^+$ a 2-sphere with constant area $4\pi \xi_0^{-2}$. This implies that the expansion of $\mathcal{I}^+$ vanishes.

- BC on $\tilde{\alpha}$ makes the time coordinate correspond to retarded Minkowski time at $\mathcal{I}^+$.

Because of the above properties, we obtain a simple expression for Bondi news function in terms of our variables (next slide):
\[ \frac{3}{K \xi_0} \tilde{\chi}_{AB} \]

- \( \tilde{\chi}_{AB} \) is the asymptotic gravitational wave amplitude.

- It is equal to the traceless part of the 2D extrinsic curvature of \( \mathcal{I}^+ \), which is a tensor that can be calculated in any coordinate system. Note that \( (A,B) \) are the angular coordinates on the 2-sphere \( \mathcal{I}^+ \).
Seek to evolve binary black holes on CMC hypersurfaces which reach future null infinity ($\mathcal{I}^+$) for high-accuracy gravitational wave modeling.

Results presented:

i) Generalized hyperboloidal Bowen-York binary black hole initial data.

ii) Bondi-Sachs energy and momentum for data on CMC slices (limited presentation to Bowen-York data).

iii) Evolution scheme.